

S. T. Chapman · P. A. García-Sánchez · D. Llena ·
A. Malyshev · D. Steinberg

On the delta set and the Betti elements of a BF-monoid

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Abstract We examine the Delta set of a cancellative and reduced atomic monoid S where every set of lengths of the factorizations of each element in S is bounded. In particular, we show the connection between the elements of $\Delta(S)$ and the Betti elements of S . We prove how the minimum and maximum element of $\Delta(S)$ can be determined using the Betti elements of S . This leads to a determination of when $\Delta(S)$ is a singleton. We then apply these results to the particular case where S is a numerical monoid that requires three generators. Conclusions are drawn in the cases where S has a unique minimal presentation, or has multiplicity three.

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المخلص

نتفحص مجموعة دلتا لمونويد حذفي ومختزل ذري S بحيث تكون مجموعة أطوال التحليلات لكل عنصر في S محدودة. نبين بشكل خاص الرابط بين عناصر $\Delta(S)$ وعناصر بيتي الخاصة بـ S . نثبت أنه يمكن تحديد العنصر الأدنى والعنصر الأعلى في $\Delta(S)$ باستخدام عناصر بيتي الخاصة بـ S . يقود هذا إلى تحديد متى تتكون $\Delta(S)$ من عنصر واحد. نطبق هذه النتائج بعد ذلك على الحالة الخاصة التي يكون S فيها مونويداً عددياً يتطلب ثلاثة مولدات. يتم استخلاص الاستنتاجات في الحالات التي يحوز فيها S تقديماً أدنى وحيداً أو تعددية تساوي ثلاثة.

S. T. Chapman (✉)
Department of Mathematics and Statistics, Sam Houston State University, Box 2206, Huntsville, TX 77341-2206, USA
E-mail: scott.chapman@shsu.edu

P. A. García-Sánchez
Departamento de Álgebra, Universidad de Granada, 18071 Granada, Spain
E-mail: pedro@ugr.es

D. Llena
Departamento de Geometría, Topología y Química Orgánica, Universidad de Almería, 04120 Almería, Spain
E-mail: dllena@ual.es

A. Malyshev
Mathematics Department, University of California at Los Angeles, Box 951555, Los Angeles, CA 90095-1555, USA
E-mail: amalyshe@princeton.edu

D. Steinberg
Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA
E-mail: djsteinbe@gmail.com



1 Introduction

Recently, several papers describing the invariants of non-unique factorizations for finitely generated commutative cancellative monoids have appeared in the literature. Many of these papers have a special emphasis on numerical monoids. For example, [8] examined properties related to the elasticity of a numerical monoid. Moreover, the papers [2, 5, 9–11] all consider properties related to the Delta set (or $\Delta(S)$) of a numerical monoid S . In particular, the results of [5] indicate that even if S is 3-generated, then the structure of $\Delta(S)$ may be extremely complex (see for example, [5, Corollary 4.8 and Proposition 4.9]). In this note, we prove some general structure theorems for $\Delta(S)$ when S is a reduced BF-monoid. Recall that S is a reduced BF-monoid if the only unit in S is its identity element, S is commutative and cancellative, and for every $s \in S$ the set of lengths of its possible factorizations is bounded. We note that the reduced assumption is not necessary, since one can always remove the units of S and consider S_{red} instead of S (see [14] for details). We then apply these results specifically to the case where S is a numerical monoid that requires three generators. This is reasonable, as several papers relating to algebraic properties of 3-generated numerical monoids have recently appeared in the literature (see [21, Chapter 9] and the references therein).

We present our results in three additional sections. In Sect. 2, we prove two structure theorems for the Delta set, and show connection between the elements of $\Delta(S)$ and the Betti elements of S . We give explicit formulas for the minimum and maximum of $\Delta(S)$. Thus we are able to determine when $\Delta(S)$ is a singleton. Conclusions are drawn in Sect. 3 in the case where S has a unique minimal presentation. We then apply these results in Sect. 4 to the particular case where S is a 3-generated numerical monoid. We consider the pseudo-symmetric, symmetric, and multiplicity 3 cases. We believe our results are of interest, as they not only improve several results from [5], but they approach these problems in a much different manner using Betti elements.

We open with some notation and definitions. Let \mathbb{N}_0 represent the natural numbers including 0. Throughout our work, we assume that all monoids are commutative, cancellative, reduced, atomic, and the set of lengths of the factorizations of every element is bounded. For known results concerning non-unique factorizations in such monoids, the interested reader is directed to the monograph [14]. For such a monoid S , there exists a set of atoms, denoted $\mathcal{A}(S)$, such that every $x \in S$ can be written in the form

$$x = c_1 n_1 + \cdots + c_t n_t, \quad (*)$$

for some $c_1, \dots, c_t \in \mathbb{N}_0, n_1, \dots, n_t \in \mathcal{A}(S)$.

We focus on the representations of elements of S in the form (*). Given $x \in S$ with $x \neq 0$, set

$$\mathcal{L}(x) = \left\{ \sum_{a \in \mathcal{A}(S)} c_a \mid x = \sum_{a \in \mathcal{A}(S)} c_a n_a, c_a \in \mathbb{N} \text{ and } n_a \in \mathcal{A}(S), \text{ for all } a \right\}$$

which is known as the *set of lengths* of x in S . We are assuming that this set of lengths is bounded, and so $\mathcal{L}(x)$ is of the form $\{m_1, m_2, \dots, m_k\}$ for some positive integers $m_1 < m_2 < \cdots < m_k$. The set

$$\Delta(x) = \{m_i - m_{i-1} \mid 2 \leq i \leq k\}$$

is known as the *Delta set* of x . We globalize the notion of the Delta set by setting

$$\Delta(S) = \bigcup_{x \neq 0 \text{ in } S} \Delta(x).$$

By a fundamental result of Geroldinger [14, Proposition 1.4.4], if $d = \gcd \Delta(S)$ and $|\Delta(S)| < \infty$, then

$$\{d\} \subseteq \Delta(S) \subseteq \{d, 2d, \dots, kd\}$$

for some $k \in \mathbb{N}$. The study of Delta sets in the class of arithmetic congruence monoids can be found in [3]. A summary of known results involving properties of Delta sets can be found in [14, Section 6.7].

If S is an additive submonoid of \mathbb{N}_0 , then S is called a *numerical monoid*. It follows from elementary number theory that S is finitely generated. If $\{n_1, \dots, n_e\}$ is a set of generators for S , then this is commonly denoted by $S = \langle n_1, \dots, n_e \rangle$. It also follows using basic techniques that the minimal generating set for S is unique. Unless otherwise noted, when dealing with a numerical monoid written as $S = \langle n_1, \dots, n_k \rangle$, we assume that $\{n_1, \dots, n_k\}$ is the minimal generating set for S . In a numerical monoid, 0 is the unique unit. A numerical monoid $S = \langle n_1, \dots, n_k \rangle$ is *primitive* if $\gcd\{n_1, \dots, n_k\} = 1$. Clearly every numerical monoid is



isomorphic to a primitive numerical monoid, hence we narrow our study to the primitive case (the interested reader is referred to [21] for more details on numerical monoids).

The following previously known results when S is a numerical monoid were the starting point of our work.

- Theorem 1.1** (1) If $S = \langle n_1, \dots, n_k \rangle$ is a primitive numerical monoid, then $\min \Delta(S) = \gcd\{n_i - n_{i-1} \mid 2 \leq i \leq k\} = \gcd \Delta(S)$ ([5, Proposition 2.9]).
- (2) If $S = \langle n, n+k, \dots, n+tk \rangle$ where $n > 1$ and $t, k \geq 1$, then $\Delta(S) = \{k\}$ ([5, Proposition 3.9] or [2, Corollary 2.3]).

Moreover, if $S = \langle n_1, n_2, n_3 \rangle$ is primitive and minimally generated, then [11, Theorem 3.1] provides a method for computing $\max \Delta(S)$ in terms of certain relations between the generators n_1, n_2 and n_3 .

2 Betti elements and Delta sets

Let S be a BF-monoid, and let $\mathbf{Z}(S) = \mathcal{F}(\mathcal{A}(S))$ the free monoid on the atoms of S . The unique monoid map $\pi_S : \mathbf{Z}(S) \rightarrow S$ that maps every $a \in \mathcal{A}(S)$ to $a \in S$ is sometimes known as the factorization homomorphism associated to S . For every $s \in S$, the set $\mathbf{Z}(s) = \pi_S^{-1}(s)$ is the set of factorizations of s .

Let \sim_S be the kernel congruence of π_S , i.e., $x \sim_S y$ if $\pi_S(x) = \pi_S(y)$, or equivalently, x and y are factorizations of the same element in S (\sim_S is actually a congruence). It follows easily that S is isomorphic to the monoid $\mathbf{Z}(S)/\sim_S$.

For $u = \sum_{a \in \mathcal{A}(S)} u_a a$ and $v = \sum_{a \in \mathcal{A}(S)} v_a a \in \mathbf{Z}(S)$, set $\gcd\{u, v\} = \sum_{a \in \mathcal{A}(S)} \min\{u_a, v_a\}a$ (this plays the role of the greatest common divisor, but with additive notation).

Given $s \in S$, we define the following binary relation on $\mathbf{Z}(s)$. For $x, y \in \mathbf{Z}(s)$, $x \mathcal{R} y$ if there exists a chain $x_1, \dots, x_k \in \mathbf{Z}(s)$ such that

- (1) $x_1 = x, x_k = y$,
- (2) $\gcd\{x_i, x_{i+1}\} \neq 0, i \in \{0, \dots, k-1\}$.

An element $s \in S$ is said to be a Betti element if $\mathbf{Z}(s)$ has more than one \mathcal{R} -class (see [12]). Observe that there are finitely many Betti elements in S if S is finitely presented. The set of Betti elements of S will be denoted by $\text{Betti}(S)$. In the finitely generated case, Betti elements are tightly related to the Betti numbers of the minimal free resolution of the semigroup ring $K[S]$, with K a field (see for instance [6]), and the elements used to construct a minimal presentation for S (we will address later).

Given $x = \sum_{a \in \mathcal{A}(S)} x_a a$ a factorization of $n \in S$, set $|x| = \sum_{a \in \mathcal{A}(S)} x_a$. Clearly, if $\mathbf{q}(\mathbf{Z}(S))$ is the group generated by $\mathbf{Z}(S)$, we can extend $|\cdot| : \mathbf{q}(\mathbf{Z}(S)) \rightarrow \mathbb{Z}$, and it is a linear map (\mathbb{Z} denotes the set of integers). With this notation $\mathcal{L}(n) = \{|x| : x \in \mathbf{Z}(n)\}$. Recall that we are assuming that this set has finitely many elements.

The following result states that one can go from one factorization to another of the same element, just using the factorizations of the Betti elements. The idea of the proof is inspired by [20, Proposition 9.3].

Lemma 2.1 Let $s \in S$ and $x, y \in \mathbf{Z}(s)$. Then there exists $z_1, \dots, z_t \in \mathbf{Z}(s)$ such that

- $z_1 = x, z_t = y$,
- for all $i \in \{1, \dots, t-1\}$, $(z_i, z_{i+1}) = (a_i + c_i, b_i + c_i)$ for some $c_i \in \mathbf{Z}(S)$ and a_i not in the same \mathcal{R} -class as b_i (and thus a_i and b_i are factorizations of a Betti element of S).

Proof If x and y are not in the same \mathcal{R} -class, then we are done. So assume that x and y are \mathcal{R} related, and let us proceed by induction on $\max \mathcal{L}(s)$ (if $\max \mathcal{L}(s) = 1$, then s is an atom, and $x = y$). By definition there exists $x_1, \dots, x_k \in \mathbf{Z}(s)$ such that $x_1 = x, x_k = y$ and $d_i = \gcd\{x_i, x_{i+1}\} \neq 0$ for all $i \in \{1, \dots, k-1\}$. Set $u_i = x_i - d_i$ and $v_i = x_{i+1} - d_i$. Notice that $u_i \sim_S v_i$, and if we define $s_i = \pi_S(u_i)$, then $\max \mathcal{L}(s_i) < \max \mathcal{L}(s)$. If u_i, v_i are in the same \mathcal{R} -class, then by the induction hypothesis there exists $z_{i_1}, \dots, z_{i_{t(i)}}$ such that $z_{i_1} = u_i, z_{i_{t(i)}} = v_i$, and $(z_{i_j}, z_{i_{j+1}}) = (a_{i_j} + k_{i_j}, b_{i_j} + k_{i_j})$ for some $k_{i_j} \in \mathbf{Z}(S)$ and a_{i_j}, b_{i_j} in different \mathcal{R} -classes. Set $c_{i_j} = d_i + k_{i_j}$. If u_i, v_i are in different \mathcal{R} -classes, then set $a_{i_1} = u_i, b_{i_1} = v_i$ and $c_{i_1} = d_i$. By putting all these sequences together, we obtain the z_1, \dots, z_t of the statement. \square

Remark 2.2 The above lemma, and thus the whole paper, can be stated in the more general setting of monoids with the ascending chain condition for principal ideals (see [14, Definition 1.1.3]). This condition is equivalent to saying that every descending divisor sequence is stationary. Observe that s_i “divides” s ($s - s_i \in S$). Hence, we can repeat the process for s_i and because of the descending divisor sequence condition, this procedure must end after a finite number of steps. Notice that by [14, Proposition 1.1.4] these monoids are always atomic.



Proposition 2.3 Let S be a BF-monoid and let $s \in S$. For every $x, y \in \mathbf{Z}(s)$, $|x| - |y|$ is of the form

$$\lambda_1 \delta_1 + \cdots + \lambda_t \delta_t,$$

for some integers $\lambda_1, \dots, \lambda_t$, and $\delta_i = ||a_i| - |b_i||$ with a_i and b_i factorizations of a Betti element in different \mathcal{R} -classes. In particular, every element in $\Delta(S)$ is of this form.

Proof As $x \sim_S y$, by Lemma 2.1, there exists a chain $x_1 = x, \dots, x_s = y$ with $(x_i, x_{i+1}) = (a_i + c_i, b_i + c_i)$, for some $c_i \in \mathbf{Z}(S)$ and a_i, b_i non- \mathcal{R} -related factorizations of a Betti element. Thus

$$|x - y| = |x_1 - x_2 + x_2 - \cdots - x_{s-1} + x_{s-1} - x_s| = \sum |a_i - b_i|,$$

and the proof follows easily. \square

Corollary 2.4 Let S be a BF-monoid. Then

$$\min \Delta(S) = \gcd\{|a - b| : a \sim_S b, a, b \text{ in different } \mathcal{R} - \text{classes}\}.$$

Proof Let $d = \min \Delta(S)$ and $d' = \gcd\{|a - b| : a \sim_S b, a, b \text{ in different } \mathcal{R} - \text{classes}\}$. In light of Proposition 2.3, $d = \lambda_1 \delta_1 + \cdots + \lambda_t \delta_t$ for some integers $\lambda_1, \dots, \lambda_t$ and $\delta_i = |a_i - b_i|$, with $a_i \sim_S b_i$ and in different \mathcal{R} -classes. Hence, d' divides d . By [17, Proposition 14], $d = \gcd\{|a - b| : (a, b) \in \mathcal{A}(\sim_S)\}$, and according to the proof of [17, Proposition 16], if $a \sim_S b$, and a and b are not in the same \mathcal{R} -class, then $(a, b) \in \mathcal{A}(\sim_S)$. Thus d divides d' , and this concludes the proof. \square

Theorem 2.5 If S is a BF-monoid with $0 < \# \Delta(S) < \infty$, then

$$\max \Delta(S) = \max_{n \in \text{Betti}(S)} (\max \Delta(n)).$$

Proof The inequality $\max_{n \in \text{Betti}(S)} (\max \Delta(n)) \leq \max \Delta(S)$ is clear, so let us focus on the other direction. Assume to the contrary that $\max \Delta(S) > \max \Delta(n)$ for all Betti elements n of S . As above, take x, y to be factorizations of an element $s \in S$ so that $|y| - |x| = \max \Delta(S)$, and consequently no other factorization z of s fulfills $|x| < |z| < |y|$. As $x \sim_S y$, let x_1, \dots, x_v be in $\mathbf{Z}(s)$ such that $x = x_1, x_v = y$ and $(x_i, x_{i+1}) = (a_i + c_i, b_i + c_i)$, with $c_i \in \mathbf{Z}(S)$, and $a_i \sim_S b_i$ and a_i, b_i in different \mathcal{R} -classes for all $i \in \{1, \dots, v-1\}$. Since $|x_1| = |x|$ and $|x_v| = |y|$, and no $|x_i|$ lies between $|x|$ and $|y|$, there exists $i \in \{1, \dots, v-1\}$, such that $|x_i| \leq |x| < |y| \leq |x_{i+1}|$. Both a_i and b_i are factorizations of an element n with $\mathbf{Z}(n)$ having more than one \mathcal{R} -class. So there is a chain of factorizations, say z_1, \dots, z_u , of n such that $a_i = z_1, \dots, z_u = b_i$, and $|z_{j+1}| - |z_j| \leq \max \Delta(n)$, which we are assuming to be smaller than $\max \Delta(S)$. But then $z_j + c_i \sim_S x \sim_S y$ for all j , and from the choice of x and y , there is no j such that $|x| < |z_j + c_i| < |y|$. Again, we can find $j \in \{1, \dots, u-1\}$ such that $|z_j + c_i| \leq |x| < |y| \leq |z_{j+1} + c_i|$. This leads to a contradiction, since $\max \Delta(S) = |y| - |x| \leq |z_{j+1} + c_i| - |z_j + c_i| = |z_{j+1} - z_j| \leq \max \Delta(n) < \max \Delta(S)$. \square

With this Theorem we get an easy characterization of finitely generated monoids with a singleton set of distances.

Corollary 2.6 The set $\Delta(S) \neq \emptyset$ is a singleton if and only if $\bigcup_{n \in \text{Betti}(S)} \Delta(n)$ is a singleton.

Observe that numerical monoids are never half factorial (the set of lengths of its elements are singletons), hence there is always a minimal relation $(a, b) \in \sigma$ with $|a| \neq |b|$, and so $\Delta(\pi_S(a))$ is not empty.

3 Finitely generated monoids having a unique minimal presentation

Let S be a monoid minimally generated by $\{n_1, \dots, n_e\}$. Given $\sigma \subseteq \mathbf{Z}(S) \times \mathbf{Z}(S)$, the congruence generated by σ is the least congruence containing σ . If \sim_S is the congruence generated by σ , then we say that σ is a presentation of S . Rédei's theorem (see [18]) precisely states that every finitely generated monoid is finitely presented. A presentation for S is a minimal presentation if none of its proper subsets generates \sim_S . In our setting, all minimal presentations have the same cardinality. Next we briefly describe a procedure for finding all minimal presentations for S as presented in [20, Chapter 9].



For every $s \in S$, define σ_s in the following way.

- If $Z(s)$ has one \mathcal{R} -class, then set $\sigma_s = \emptyset$.
- Otherwise, let $\mathcal{R}_1, \dots, \mathcal{R}_k$ be the different \mathcal{R} -classes of $Z(s)$. Choose any $a_i \in \mathcal{R}_i$ for all $i \in \{1, \dots, k\}$ and set σ_s to be any set of $k-1$ pairs of elements in $V = \{a_1, \dots, a_k\}$ so that any two elements in V are connected by a sequence of pairs in σ_s (or their symmetric). For instance, we can choose $\sigma_s = \{(a_1, a_2), \dots, (a_1, a_k)\}$.

Then $\sigma = \bigcup_{s \in S} \sigma_s$ is a minimal presentation of S . Moreover, in this way one can construct all minimal presentations for S .

We say that S has a unique minimal presentation if for any other minimal presentation σ' , and any relator $(a, b) \in \sigma'$, then either $(a, b) \in \sigma$ or $(b, a) \in \sigma$ (i.e., σ is unique up to rearrangement of the components of its relators). Monoids having a unique minimal presentation have drawn the attention of many researchers in the last decade (see for instance [12]).

Corollary 3.1 Assume that S is uniquely presented with $\Delta(S) \neq \emptyset$, and that its unique minimal presentation is $\{(a_1, b_1), \dots, (a_t, b_t)\}$. If we set $\delta_i = ||a_i| - |b_i||$, then

$$\min \Delta(S) = \gcd\{\delta_1, \dots, \delta_t\} \quad \text{and} \quad \max \Delta(S) = \max\{\delta_1, \dots, \delta_t\}.$$

Moreover, $\Delta(S)$ is a singleton if and only if $\{0\} \neq \{\delta_1, \dots, \delta_t\} \subseteq \{0, \gcd\{\delta_1, \dots, \delta_t\}\}$.

Proof The first equality follows from Corollary 2.4, since every element with more than one \mathcal{R} -class has only two factorizations, and these occur in the minimal presentation of S .

As a consequence of [12, Corollaries 5 and 6], every Betti element in S has just two factorizations (and each is in a different \mathcal{R} -class). Hence, $\text{Betti}(S) = \{s_1, \dots, s_t\}$ and for all $i \in \{1, \dots, t\}$, $Z(s_i) = \{a_i, b_i\}$. This, in particular, means that $\Delta(s_i) \subseteq \{\delta_i\}$, for all i , and $\max \Delta(S) = \max\{\delta_1, \dots, \delta_t\}$. Clearly, $\Delta(S)$ is a singleton if and only if $\{0\} \neq \{\delta_1, \dots, \delta_t\} \subseteq \{0, \gcd\{\delta_1, \dots, \delta_t\}\}$ (notice that by [22, Proposition 22], not all δ_i can be zero). \square

We now recall the definition of another invariant that is tightly related to $\max \Delta(S)$. If $s \in S$, $z, z' \in Z(s)$, and N is a non-negative integer, then an N -chain of factorizations from z to z' is a sequence $z_0, \dots, z_k \in Z(s)$ such that $z_0 = z$, $z_k = z'$ and $\max\{|z_i - d_i|, |z_{i+1} - d_i|\} \leq N$ for all i , and $d_i = \gcd(z_i, z_{i+1})$. The catenary degree of S , $\mathbf{c}(S)$, is the minimal $N \in \mathbb{N}_0 \cup \{\infty\}$ such that for any $s \in S$ and any two factorizations $z, z' \in Z(s)$, there is an N -chain from z to z' . It is well known ([14, Proposition 1.6.3]) that

$$\sup \Delta(S) + 2 \leq \mathbf{c}(S).$$

Observe also that if S is a uniquely presented finitely generated monoid, then as a consequence of [7, Theorem 3.1]

$$\mathbf{c}(S) = \max\{\max\{|a_i|, |b_i|\} : i \in \{1, \dots, t\}\}.$$

4 Embedding dimension three numerical monoids

Let us recall some basic facts concerning numerical monoids in general. Define

$$T(S) = \{g \in \mathbb{Z} \setminus S \mid g + S \setminus \{0\} \subseteq S\}.$$

The cardinality of this set is the type of S . Observe that the largest integer not belonging to S , its Frobenius number denoted by $F(S)$, is always in $T(S)$ (recall that we are assuming that S is primitive, and thus $\mathbb{N} \setminus S$ is always finite). Numerical monoids of type one are symmetric (or equivalently, the cardinality of $\mathbb{N} \setminus S$ equals $\frac{F(S)+1}{2}$; see [21] for more details), and numerical monoids with $T(S) = \{F(S), F(S)/2\}$ are called pseudo-symmetric. For an element $n \in S$, define its Apéry set as $\text{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. This set contains exactly n elements, each being the minimum in its congruence class modulo n .

We now restrict to embedding dimension three numerical monoids. In this setting, minimal presentations either have cardinality two or three, and the shape of these presentations are well known (see for instance [21, Chapter 9]). Numerical monoids having just a couple of relations are complete intersections (both free and symmetric). We show that the sets of distances can be easily described in this setting.



4.1 Generic case, embedding dimension three

Let S be an embedding dimension three numerical monoid. Assume that $\{n_1 < n_2 < n_3\}$ is its set of minimal generators, and that S is not symmetric. Let c_i, r_{ij} be as in [21, Example 8.23], i.e.,

$$c_i n_i = r_{ij} n_j + r_{ik} n_k,$$

with $\{i, j, k\} = \{1, 2, 3\}$ and $c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle\}$. As S is not symmetric, we have $r_{ij} > 0$ (see [15] or [21, Chapter 9]). Moreover, this presentation is generic in the sense of Peeva and Sturmfels [16], and thus S is uniquely presented. In other words, r_{ij} are unique (see for instance [4, Proposition 5.5]). This, in particular, means that $r_{ij} \leq c_i$ for all i, j . Hence, we can copy/paste [21, Lemma 10.19] and obtain that for $\{i, j, k\} = \{1, 2, 3\}$,

$$c_i = r_{ji} + r_{ki}.$$

Hence, we always have that

$$(c_1, -r_{12}, -r_{13}) + (-r_{21}, c_2, -r_{23}) + (-r_{31}, -r_{32}, c_3) = (0, 0, 0). \quad (1)$$

Observe that

$$\sigma = \{((c_1, 0, 0), (0, r_{12}, r_{13})), ((0, c_2, 0), (r_{21}, 0, r_{23})), ((0, 0, c_3), (r_{31}, r_{32}, 0))\}$$

is the (unique) minimal presentation of S .

In our setting, $\delta_i = |c_i - (r_{ij} + r_{ik})|$ for $\{i, j, k\} = \{1, 2, 3\}$. Thus $\delta_1 = c_1 - r_{12} - r_{13}$ is a positive integer ($c_1 n_1 = r_{12} n_2 + r_{13} n_3 > r_{12} n_1 + r_{13} n_1$), $\delta_2 = |c_2 - r_{21} - r_{23}| = |\delta_1 - \delta_3|$, and $\delta_3 = r_{31} + r_{32} - c_3 > 0$. Hence, Proposition 2.3 implies that every element in $\Delta(S)$ is of the form

$$\lambda_1 \delta_1 + \lambda_2 \delta_2.$$

We already know from Corollary 3.1 that $\max \Delta(S) = \max\{\delta_1, \delta_2, \delta_3\}$, and as $\delta_2 = |\delta_1 - \delta_3|$, we obtain

$$\max \Delta(S) = \max\{\delta_1, \delta_3\}.$$

Observe that $\Delta(c_1 n_1) = \{\delta_1\}$, $\Delta(c_3 n_3) = \{\delta_3\}$, and either $\Delta(c_2 n_2) = \emptyset$ (when $\delta_2 = 0$) or $\Delta(c_2 n_2) = \{\delta_2\}$ (if $\delta_2 \neq 0$). Hence, $\Delta(S) = \{d\}$ if and only if $\delta_1 = \delta_3 = d$ and $\delta_2 = 0$, or equivalently, $\delta_2 = 0$. Thus, $\Delta(S)$ is a singleton if and only if $\delta_2 = 0$.

Notice that if $\Delta(S)$ is a singleton, then

$$n_1 = \frac{r_{12}(n_2 - n_1) + r_{13}(n_3 - n_1)}{d}, \quad n_3 = \frac{r_{31}(n_3 - n_1) + r_{32}(n_3 - n_2)}{d}.$$

As we mentioned above, $\mathbf{c}(S) = \max\{c_1, r_{12} + r_{13}, c_2, r_{21} + r_{23}, c_3, r_{31} + r_{32}\}$ (and also equal to $\mathbf{t}(S)$, the tame degree of S , and $\omega(S)$, the ω -primality of S ; see [4] for details). Since $c_1 > r_{12} + r_{13}$ and $c_3 < r_{31} + r_{32}$, this maximum is $\max\{c_1, c_2, r_{21} + r_{23}, r_{31} + r_{32}\}$. See [1] for a different approach to the computation of the catenary degree of embedding dimension three numerical semigroups.

4.2 Pseudo-symmetric case

This is a subcase of the generic case. According to [21, Proposition 10.13], the minimal generators of S can be arranged so that

$$\begin{aligned} c_1 n_1 &= (c_2 - 1)n_2 + n_3, \\ c_2 n_2 &= (c_3 - 1)n_3 + n_1, \\ c_3 n_3 &= (c_1 - 1)n_1 + n_2, \end{aligned}$$

and a minimal presentation (actually the unique minimal presentation) of S is

$$\sigma = \{((c_1, 0, 0), (0, c_2 - 1, 1)), ((0, c_2, 0), (1, 0, c_3 - 1)), ((0, 0, c_3), (c_1 - 1, 1, 0))\}.$$

With this we obtain the following.



Theorem 4.1 Let $S = \langle n_1, n_2, n_3 \rangle$ be an embedding dimension three pseudo-symmetric numerical monoid. For every $\{i, j, k\} = \{1, 2, 3\}$ define $c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle\}$. Then

$$\max \Delta(S) = \max\{|c_i - c_j| \mid 1 \leq i < j \leq 3\},$$

and $\Delta(S)$ is a singleton if and only if $\{|c_i - c_j| \mid 1 \leq i < j \leq 3\} = \{0, d\}$.

In this setting $\mathbf{c}(S) = \mathbf{t}(S) = \omega(S) = \max\{c_1, c_2, c_3\}$.

Recall that $\max \Delta(S) + 2 \leq \mathbf{c}(S)$ (in our scope the supremum in [14, Theorem 1.6.3] becomes a maximum). Let us see when the equality $\max \Delta(S) + 2 = \mathbf{c}(S)$ holds for an embedding dimension three pseudo-symmetric numerical semigroup. Assume that $c_1 \geq c_2 \geq c_3$ (the rest of cases are studied analogously). Then $\mathbf{c}(S) = c_1 = \max\{c_1 - c_2, c_1 - c_3, c_2 - c_3\} + 2$. Hence, $c_1 \in \{c_1 - c_2 + 2, c_1 - c_3 + 2, c_2 - c_3 + 2\}$.

- Observe that if $c_1 = c_2 - c_3 + 2$, then $c_1 + c_3 = c_2 + 2$. As $\min\{c_1, c_2, c_3\} \geq 2$, since $\{n_1, n_2, n_3\}$ is a minimal system of generators, we get that $c_1 + c_3 = c_2 + 2$ if and only if $c_1 = c_2$; $c_3 = 2$.
- If $c_1 = c_1 - c_2 + 2$, then we get $c_2 = 2$, which forces c_3 to be also equal to 2.
- Finally, if $c_1 = c_1 - c_3 + 2$, then we obtain $c_3 = 2$.

Hence, if $\max \Delta(S) + 2 = \mathbf{c}(S)$, then $c_3 = 2$. Using [21, Corollary 10.15], we obtain that $n_1 = c_3(c_2 - 1) + 1$, $n_2 = c_1(c_3 - 1) + 1$, and $n_3 = c_2(c_1 - 1) + 1$. Thus,

$$S = \langle 2c_2 - 1, c_1 + 1, c_2(c_1 - 1) + 1 \rangle,$$

with $c_1 \geq c_2 \geq 2$. Notice that if in addition $c_2 = 2$, then $n_1 = 3$ (these semigroups are studied in the last section). The reader is referred to [13, Corollary 4.3] for other families of monoids for which the equality $\max \Delta(S) + 2 = \mathbf{c}(S)$ holds.

4.3 Symmetric case, embedding dimension three

Let S be an embedding dimension three symmetric numerical monoid (and thus a free and complete intersection). According to [21, Theorem 10.6], $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$, for some non-negative integers m_1, m_2, a, b, c such that

- $a \geq 2, b + c \geq 2$,
- $\gcd\{m_1, m_2\} = 1 = \gcd\{a, bm_1 + cm_2\}$.

A minimal presentation for S (uniqueness is only granted when $0 < b < m_2$ and $0 < c < m_1$, see [12, Theorem 17]) is

$$\sigma = \{((m_2, 0, 0), (0, m_1, 0)), ((0, 0, a), (b, c, 0))\}.$$

Assume without loss of generality that $m_1 < m_2$. Hence, $\delta_1 = m_2 - m_1 > 0$ and $\delta_2 = |a - b - c|$.

Notice that the set of factorizations of $m_1 m_2$ is $\{(m_2, 0, 0), (0, m_1, 0)\}$ and that if r and s are non-negative integers such that $rm_1 \leq c$, $(r + 1)m_1 > c$, $sm_2 \leq b$ and $(s + 1)m_2 > b$, then the set of factorizations of $a(bm_1 + cm_2)$ is

$$\{(0, 0, a), (b - sm_2, c + sm_1, 0), \dots, (b - m_2, c + m_1, 0), (b, c, 0), \\ (b + m_2, c - m_1, 0), \dots, (b + rm_2, c - rm_1, 0)\}.$$

The set of lengths of $a(bm_1 + cm_2)$ is $\{a, b + c - s(m_2 - m_1), \dots, b + c, \dots, b + c + r(m_2 - m_1)\}$. If $\Delta(S) = \{d\}$, then $m_2 - m_1 = d$, and either

- $a < b + c - sd$ and $b + c - sd - a = d$ (equivalently, $a = b + c - (s + 1)d$), or
- $a = b + c + kd$ for some $k \in \{-s, \dots, r\}$, or
- $b + c + rd < a$ and $a - b - c - rd = d$ (equivalently, $a = b + c + (r + 1)d$).

Observe that the converse is also true. That is, $\Delta(S)$ is a singleton if and only if $m_2 - m_1 = d$ and $a = b + c + kd$ for some $k \in \{-s - 1, \dots, r + 1\}$.

Notice also that if $b + c - s(m_2 - m_1) - a > m_2 - m_1$, then $\max \Delta(S) = b + c - s(m_2 - m_1) - a$, and if $a - (b + c + r(m_2 - m_1)) > m_2 - m_1$, $\max \Delta(S) = a - (b + c + r(m_2 - m_1))$. And in any other case, $\max \Delta(S) = m_2 - m_1$.



4.4 Multiplicity three

Assume that S is minimally generated by $\{3 < n_2 < n_3\}$. As the Apéry set of 3 in S is $\{0, n_2, n_3\}$, by Selmer's formula (see for instance [21, Proposition 2.10]), we have that

- $F(S) = n_3 - 3$ (the Frobenius number of S), and
- $G(S) = \frac{1}{3}(n_2 + n_3) - 1$ (the genus of S , i.e., the cardinality of $\mathbb{N} \setminus S$).

Hence, S is of the form $\langle 3, 3G(S) - F(S), F(S) + 3 \rangle$ (see [19] for more details).

First we prove that S cannot be symmetric. If S is symmetric, then $G(S) = \frac{F(S)+1}{2}$ and consequently $n_2 = 3G(S) - F(S) = \frac{3}{2}(F(S) + 1) - F(S) = \frac{1}{2}(F(S) + 3) = \frac{1}{2}n_3$. This contradicts that $\{3, n_2, n_3\}$ is a minimal system of generators.

As r_{ij} are positive integers, Lemmas 10.19 to 10.23 in [21] hold, and consequently $3 = r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{32}$, forcing $r_{12} = r_{13} = r_{23} = r_{32} = 1$. Hence, $\delta_1 = \frac{n_2+n_3}{3} - 2$. Moreover, $r_{21} + r_{31} = c_1$, $r_{12} + r_{32} = c_2$, $r_{13} + r_{23} = c_3$. This implies that $c_1 = \frac{n_2+n_3}{3}$, $c_2 = 2$ and $c_3 = 2$. Hence, σ has the following form

$$\sigma = \left\{ \left(\left(\frac{n_2+n_3}{3}, 0, 0 \right), (0, 1, 1) \right), ((0, 2, 0), (a, 0, 1)), \left((0, 0, 2), \left(\frac{n_2+n_3}{3} - a, 1, 0 \right) \right) \right\}.$$

Note that $\delta_1 = \frac{n_2+n_3}{3} - 2 = r_{31} + r_{21} - 2 \geq r_{31} + 1 - 2 = \delta_3$. Then $\delta_2 = |\delta_1 - \delta_3| = \delta_1 - \delta_3$, and $\delta_1 = \delta_2 + \delta_3 \geq \delta_2$. Hence, we get the following.

Theorem 4.2 *Let $S = \langle 3, n_2, n_3 \rangle$ be an embedding dimension three numerical monoid. Then*

$$\max \Delta(S) = \frac{n_2 + n_3}{3} - 2.$$

Notice also that as S has a generic presentation, $\mathbf{c}(S) = \mathbf{t}(S) = \omega(S) = \max\{\frac{n_2+n_3}{3}, 2, a+1, \frac{n_2+n_3}{3} - a + 1\} = \frac{n_2+n_3}{3}$. Hence, we can also deduce the above theorem by taking into account that $\max \Delta(S) + 2 \leq \mathbf{c}(S)$ ([14, Theorem 1.6.3]), and $\mathcal{L}(n_2 + n_3) = \{2, (n_2 + n_3)/2\}$.

Theorem 4.3 *Let $S = \langle 3, n_2, n_3 \rangle$ be an embedding dimension three numerical monoid. The following are equivalent.*

- (1) $\Delta(S)$ is a singleton.
- (2) $S = \langle 3, x, 2x - 3 \rangle$ for some integer x greater than three.
- (3) $G(S) = \frac{F(S)+2}{2}$.

Proof Recall that $\Delta(S)$ is a singleton if and only if $\delta_2 = 0$, which in our case is equivalent to $a + 1 - 2 = 0$. Hence, $\Delta(S)$ is a singleton if and only if $a = 1$. The last condition is equivalent to $2n_2 = 3 + n_3$ (because there are only two factorizations of $2n_2$, say $(0, 2, 0)$ and $(a, 0, 1)$). This proves that (1) and (2) are equivalent. The equivalence between (2) and (3) follows from [19, Theorem 7] (in this setting $F(S) = 2x - 6$). \square

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